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Memorandum № 1

Keplerian Orbit Elements ⟶ Cartesian State Vectors

Inputs

- a traditional set of Keplerian Orbit Elements
  - Semi-major axis a [m]
  - Eccentricity e [1]
  - Argument of periapsis ω [rad]
  - Longitude of ascending node (LAN) Ω [rad]
  - Inclination i [rad]
  - Mean anomaly $M_0 = M(t_0)$ [rad] at epoch t_0 [JD]
- considered epoch t [JD], if different from t_0
- standard gravitational parameter $\mu = GM$ of the central body, if different from Sun (G...Newtonian constant of gravitation $[\frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}]$, M...central body mass [kg])

Outputs

- cartesian state vectors
  - position vector $\mathbf{r}(t)$ [m] or [AU]
  - velocity vector $\dot{\mathbf{r}}(t)$ $[\frac{\mathrm{m}}{\mathrm{s}}]$ or $[\frac{\mathrm{AU}}{\mathrm{d}}]$

1 Algorithm

1. Calculate or set $M(t)$:
   a) If $t = t_0$: $M(t) = M_0$.
   b) If $t \neq t_0$:[1]
      i. Determine the time difference Δt in seconds with

$$\Delta t = 86\,400(t - t_0). \tag{1}$$

      ii. Calculate mean anomaly $M(t)$ from

$$M(t) = M_0 + \Delta t\sqrt{\frac{\mu}{a^3}} \tag{2}$$

with $\mu = \mu_{\odot} = 1.327\,124\,400\,41 \cdot 10^{20}\ (\pm 1 \cdot 10^{10})\ \frac{\mathrm{m}^3}{\mathrm{s}^2}$ for the Sun as central body. Normalize $M(t)$ to be in $[0, 2\pi)$.

2. Solve Kepler's Equation $M(t) = E(t) - e \sin E$ for the eccentric anomaly $E(t)$ with an appropriate method numerically, e.g. the Newton–Raphson method[2]:

$$f(E) = E - e\sin E - M \tag{3}$$

$$E_{j+1} = E_j - \frac{f(E_j)}{\frac{\mathrm{d}}{\mathrm{d}E_j} f(E_j)} = E_j - \frac{E_j - e\sin E_j - M}{1 - e\cos E_j}, \quad E_0 = M \tag{4}$$

3. Obtain the true anomaly $\nu(t)$ from

$$\nu(t) = 2 \cdot \operatorname{arctan2}\left(\sqrt{1+e}\sin\frac{E(t)}{2}, \sqrt{1-e}\cos\frac{E(t)}{2}\right), \tag{5}$$

[1] Be aware that Orbit Elements change over time, so be sure to use one set of Orbit Elements given for a certain epoch t_0 only for a small time interval (compared to the rate of changes of the Orbit Elements) around t_0.
[2] Argument (t) omitted for the sake of simplicity.

Errors, comments or ideas regarding this paper? → https://goo.gl/XB85LZ where arctan2 is the two-argument arctangent function

$$\operatorname{arctan2}(y,x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & y \geq 0, x < 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & y < 0, x < 0 \\ +\frac{\pi}{2} & y > 0, x = 0 \\ -\frac{\pi}{2} & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0 \end{cases}. \tag{6}$$

4. Use the eccentric anomaly $E(t)$ to get the distance to the central body with

$$r_c(t) = a(1 - e\cos E(t)). \tag{7}$$

5. Obtain the position and velocity vector $\mathbf{o}(t)$ and $\dot{\mathbf{o}}(t)$, respectively, in the orbital frame (z-axis perpendicular to orbital plane, x-axis pointing to periapsis of the orbit):

$$\mathbf{o}(t) = \begin{pmatrix} o_x(t) \\ o_y(t) \\ o_z(t) \end{pmatrix} = r_c(t) \begin{pmatrix} \cos\nu(t) \\ \sin\nu(t) \\ 0 \end{pmatrix}, \quad \dot{\mathbf{o}}(t) = \begin{pmatrix} \dot{o}_x(t) \\ \dot{o}_y(t) \\ \dot{o}_z(t) \end{pmatrix} = \frac{\sqrt{\mu a}}{r_c(t)} \begin{pmatrix} -\sin E \\ \sqrt{1-e^2}\cos E \\ 0 \end{pmatrix} \tag{8}$$

6. Transform $\mathbf{o}(t)$ and $\dot{\mathbf{o}}(t)$ to the inertial frame[3] in bodycentric (in case of the Sun as central body: heliocentric) rectangular coordinates $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$ with the rotation matrices $\underline{R}_x(\varphi)$ and $\underline{R}_z(\varphi)$ using the transformation sequence

$$\begin{aligned} \mathbf{r}(t) &= \underline{R}_z(-\Omega)\underline{R}_x(-i)\underline{R}_z(-\omega)\mathbf{o}(t) \\ &\overset{o_z(t)=0}{=\!=\!=} \begin{pmatrix} o_x(t)(\cos\omega\cos\Omega - \sin\omega\cos i\sin\Omega) - o_y(t)(\sin\omega\cos\Omega + \cos\omega\cos i\sin\Omega) \\ o_x(t)(\cos\omega\sin\Omega + \sin\omega\cos i\cos\Omega) + o_y(t)(\cos\omega\cos i\cos\Omega - \sin\omega\sin\Omega) \\ o_x(t)(\sin\omega\sin i) + o_y(t)(\cos\omega\sin i) \end{pmatrix} \end{aligned} \tag{9}$$

$$\begin{aligned} \dot{\mathbf{r}}(t) &= \underline{R}_z(-\Omega)\underline{R}_x(-i)\underline{R}_z(-\omega)\dot{\mathbf{o}}(t) \\ &\overset{\dot{o}_z(t)=0}{=\!=\!=} \begin{pmatrix} \dot{o}_x(t)(\cos\omega\cos\Omega - \sin\omega\cos i\sin\Omega) - \dot{o}_y(t)(\sin\omega\cos\Omega + \cos\omega\cos i\sin\Omega) \\ \dot{o}_x(t)(\cos\omega\sin\Omega + \sin\omega\cos i\cos\Omega) + \dot{o}_y(t)(\cos\omega\cos i\cos\Omega - \sin\omega\sin\Omega) \\ \dot{o}_x(t)(\sin\omega\sin i) + \dot{o}_y(t)(\cos\omega\sin i) \end{pmatrix}. \end{aligned} \tag{10}$$

7. In order to obtain the position and velocity vector $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$, respectively, in the units AU and AU/d, calculate

$$\mathbf{r}(t)_{[\mathrm{AU}]} = \frac{\mathbf{r}(t)}{1.495\,978\,706\,91 \cdot 10^{11}}, \qquad \dot{\mathbf{r}}(t)_{[\mathrm{AU/d}]} = \frac{\dot{\mathbf{r}}(t)}{86\,400 \cdot 1.495\,978\,706\,91 \cdot 10^{11}}. \tag{11}$$

2 Constants and Conversion Factors

Universal Constants

Symbol	Description	Value	Source
G	Newtonian constant of gravitation[4]	$G = 6.67428(67) \cdot 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}$	[2, pp. 686–689]

Conversion Factors

Conversion		Source
Astronomical Units ⟶ Meters	$1\ \mathrm{AU} = 1.495\,978\,707\,00 \cdot 10^{11}\ (\pm 3)\ \mathrm{m}$	[4, p. 370 f.]
Julian Days ⟶ Seconds	$1\ \mathrm{d} = 86\,400\ \mathrm{s}$	[5, p. 696]
Degrees ⟶ Radians	$1^\circ = 1^\circ \cdot \frac{\pi}{180^\circ}\ \mathrm{rad} \approx 0{,}017453293\ \mathrm{rad}$	

[3] W.r.t. the central body and the meaning of i, ω and Ω to its reference frame.
[4] The numbers in parentheses in $6.67428(67) \cdot 10^{-11}$ are a common way to state the uncertainty; short notation for $(6.67428 \pm 0.0000067) \cdot 10^{-11}$.

3 References

Equations 2–4, 7 and 8: [3, pp. 22–27]; Equations 9 and 10: [6, p. 26]; Equation 5: [7]; Value for $\mu_{\odot}$: [1].

[1] IAU Division I Working Group on Numerical Standards for Fundamental Astronomy: *Astronomical Constants: Current Best Estimates (CBEs)*. Online available at http://maia.usno.navy.mil/NSFA/NSFA_cbe.html (retrieved 2017/10/05). [⟶ cited on page 3]

[2] Mohr, Peter J.; Taylor, Barry N.; Newell, David B.: *CODATA recommended values of the fundamental physical constants: 2006*. In: *Review of Modern Physics* **80** (2): 633–730. American Physical Society, 2008. ISSN: 1539-0756. DOI: 10.1103/RevModPhys.80.633. [⟶ cited on page 2]

[3] Montenbruck, Oliver; Gill, Eberhard: *Satellite Orbits: Models, Methods, Applications*. Corrected 3rd printing. Springer, Heidelberg, 2005. ISBN: 9783540672807. [⟶ cited on page 3]

[4] Pitjeva, E.; Standish, E.: *Proposals for the masses of the three largest asteroids, the Moon-Earth mass ratio and the Astronomical Unit*. In: *Celestial Mechanics and Dynamical Astronomy* **103** (4): 365–372. Springer Netherlands, 2009. ISSN: 0923-2958. DOI: 10.1007/s10569-009-9203-8. [⟶ cited on page 2]

[5] Seidelmann, P. Kenneth (ed.): *Explanatory Supplement to the Astronomical Almanac*. First paperback impression. University Science Books, Sausalito, California, USA, 2006. ISBN: 978-1-891389-45-0. [⟶ cited on page 2]

[6] Standish, E. Myles; Williams, James G.: *Orbital Ephemerides of the Sun, Moon and Planets*. Online available at ftp://ssd.jpl.nasa.gov/pub/eph/planets/ioms/ExplSupplChap8.pdf (retrieved 2011/05/15). [⟶ cited on page 3]

[7] Wikipedia (ed.): *True anomaly*. Online available at http://en.wikipedia.org/w/index.php?title=True%5C_anomaly&oldid=427251318 (retrieved 2011/05/15). [⟶ cited on page 3]